

Spherically Symmetric Random Permutations

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Abstract

We consider random permutations which are spherically symmetric with respect to a metric on the symmetric group S_n and are consistent as n varies. The extreme infinitely spherically symmetric permutation-valued processes are identified for the Hamming, Kendall-tau and Caley metrics. The proofs in all three cases are based on a unified approach through stochastic monotonicity.

MSC:

Keywords: random permutations, spherical symmetry, Martin boundary.

1 Introduction

Characterisation of processes with symmetries as mixtures of extreme processes is a central theme in the circle of ideas surrounding de Finetti's theorem on infinite exchangeability. A distinguished example is Freedman's [9] representation of a spherically symmetric sequence of real random variables ξ_1, ξ_2, \dots as a scale mixture of i.i.d. zero-mean Gaussian sequences. This result is equivalent to Schoenberg's theorem from analysis, see [4, 18, 21] for background and various proofs.

Traditionally, spherical symmetry in n dimensions is defined as invariance of the distribution of ξ_1, \dots, ξ_n under the group of orthogonal transformations. But this property holds precisely when the conditional distribution on every sphere centred at the origin is uniform. The latter interpretation is better suitable for generalisation to metric spaces other than Euclidean and, in fact, this kind of extension of Freedman's theorem for L^p -spherically symmetric sequences ξ_1, ξ_2, \dots has appeared in the literature [4]. Seeking for further analogues one is naturally lead to consider the infinite spherical symmetry in the framework of projective limits of metric spaces, as counterparts of the real space \mathbb{R}^∞ .

In this paper we explore the setting of combinatorial spaces of permutations S_n equipped with some metric. There are many meaningful metrics on S_n used in applications to quantify the unsortedness of permutation or the similarity between rankings

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[7], but it is far from obvious if these can be complemented by projections that preserve the spherical symmetry. We observe that such projections connecting the S_n 's exist for three classic metrics – Hamming, Kendall-tau and Cayley – and for each of these we identify explicitly the extreme permutation-valued processes with spherical symmetry. On the technical side, we will emphasize the approach based on stochastic monotonicity. This method has been previously used in [12] in a setting closely related to ours and in [6] in the study of Markov chains on the Young graph arising in the asymptotic representation theory of symmetric groups.

2 Virtual permutations

Suppose S_n ($n = 1, 2, \dots$), the symmetric group of permutations of $[n] = \{1, \dots, n\}$, is equipped with some metric. Let $|\pi|$ denote the distance between $\pi \in S_n$ and the identity permutation, so $\{\pi \in S_n : |\pi| = r\}$ is a sphere of radius r centred at the identity.

A random permutation Π of $[n]$ is just a random variable with values in S_n . We say that Π is *spherically symmetric* if the probability $\mathbb{P}(\Pi = \pi)$ depends on $\pi \in S_n$ only through $|\pi|$. For the family of spherically symmetric permutations, the random variable $|\Pi|$ is a sufficient statistic, in the sense that given $|\Pi| = r$ the conditional distribution of Π is uniform on the sphere of radius r .

Let $f_n : S_n \rightarrow S_{n-1}$ be a system of n -to-1 projections ($n > 1$). Wherever $f_n(\pi) = \sigma$ for $\pi \in S_n$ and $\sigma \in S_{n-1}$ we say that σ is the projection of π , and that π is an extension of σ . Thus every $\sigma \in S_{n-1}$ has exactly n extensions in S_n . Generalising by induction this relation, we define for $\nu > n$ the projection $f_{\nu \downarrow n} : S_\nu \rightarrow S_n$ through $f_{\nu \downarrow n} := f_{n+1} \circ f_{n+2} \circ \dots \circ f_\nu$. For $\sigma \in S_n$ and $\pi \in S_\nu$ with $n < \nu$ we say that π is an extension of σ (and σ is the projection of π) if $f_{\nu \downarrow n}(\pi) = \sigma$.

We shall assume throughout that the metric is consistent with the projections, meaning that for all $n > 1, r \geq 0$ and $\sigma \in S_{n-1}$ the number of extensions $\#\{\pi \in S_n : |\pi| = r, f_n(\pi) = \sigma\}$ depends on σ only through $|\sigma|$. The condition is needed to ensure that spherical symmetry is preserved by the projections, indeed the consistency entails that

$$\mathbb{P}(f_n(\Pi) = \sigma) = \sum_{r \geq 0} \sum_{\pi \in S_n : |\pi| = r, f_n(\pi) = \sigma} \mathbb{P}(\Pi = \pi) \quad (2.1)$$

only depends on $|\sigma|$.

The projective limit of (S_n, f_n) 's is the compact (in the product topology) space of sequences $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots)$ with $\pi_n \in S_n$ and $f_n(\pi_n) = \pi_{n-1}$ for $n > 1$. We call $\boldsymbol{\pi}$ a *virtual permutation*. The term is borrowed from [19], where a particular projective limit of the permutation spaces was considered. In known examples a virtual permutation can be interpreted as a kind of combinatorial structure build upon the infinite set \mathbb{N} , either a bijection (infinite permutation) $\mathbb{N} \rightarrow \mathbb{N}$ or a more complex object.

A random virtual permutation $\boldsymbol{\Pi} = (\Pi_1, \Pi_2, \dots)$ is a random variable, which we canonically realize as the identity function on the projective limit space endowed with some probability measure (distribution of $\boldsymbol{\Pi}$). By the measure extension theorem, the distribution of $\boldsymbol{\Pi}$ is uniquely determined by the marginal distributions of Π_n 's, provided these are

consistent with the projections. We may also view Π dynamically as a permutation-valued growth process, where Π_n extends Π_{n-1} in some random fashion.

Special notation $\Pi^* = (\Pi_1^*, \Pi_2^*, \dots)$ will be used for the *uniform* virtual permutation which has Π_n^* uniformly distributed over S_n for every n ; the consistency in this case follows from the n -to-1 property of f_n 's. To construct Π^* sequentially, at each step the extension must be chosen from the available options uniformly. It should be stressed that the support and distribution of Π^* depend on the type of projections.

In the sequel we focus on *infinitely spherically symmetric* (ISS) random virtual permutations Π , which have every $\Pi_n, n = 1, 2, \dots$, spherically symmetric. Clearly, the uniform Π^* is ISS.

The sequence of sufficient statistics $|\Pi| := (|\Pi_1|, |\Pi_2|, \dots)$ is a Markov chain naturally associated with ISS virtual permutation. The Markov property for $|\Pi|$ is readily justified by looking at the time-reversed process. Moreover, the backward transition probabilities for $|\Pi|$ do not depend on particular Π , hence are the same as for the uniform Π^* . Conversely, if a time-inhomogeneous Markov chain at every step has the same range and the same backward transition probabilities as $|\Pi^*|$, then it can be uniquely realised as $|\Pi|$ for some ISS virtual permutation.

The set of spherically symmetric distributions for each Π_n is a simplex. The family of ISS virtual permutations is a projective limit of these finite-dimensional simplices, thus by a general result (see e.g. [15], p. 164) it is a Choquet simplex, i.e. a convex compact set with the property that each element has a unique representation as a mixture of extreme elements. In this sense the problem of describing the ISS virtual permutations amounts to identifying the extremes.

There is one very general approach to the problem, which can be traced back deeply in history. By a theorem attributed to Maxwell and Borel [18], the projection to n dimensions of the uniform distribution on a sphere in \mathbb{R}^ν of radius $\lambda\nu^{1/2}$ converges, as $\nu \rightarrow \infty$, to the product of n copies of $\mathcal{N}(0, \lambda^2)$. Comparing with Freedman's theorem, it follows that all extreme ISS sequences in the Euclidean setting appear as limits of such projections from spheres in high dimensions. Likewise, let $\mathfrak{U}_{\nu,r}$ be a uniformly random element of the sphere of radius r in the symmetric group S_ν . Restating another general result (cf. [8], Theorem 4.1) we have the following analogue.

Theorem 2.1. *If $\Pi = (\Pi_1, \Pi_2, \dots)$ is an extreme ISS virtual permutation, then there exists a sequence of numbers $r(\nu)$, $\nu = 1, 2, \dots$ such that for each $n = 1, 2, \dots$ the sequence $f_{\nu \downarrow n}(\mathfrak{U}_{\nu, r(\nu)})$ converges in distribution to Π_n as $\nu \rightarrow \infty$.*

The set of probability distributions that arise as such limits is called, depending on the context, the Martin boundary or the family of Boltzmann laws.

Here is another simple property of the Euclidean spheres, which can be used to give yet another proof of Friedman's theorem. Let U be uniformly distributed on the unit sphere in \mathbb{R}^ν , and let ρ_n be the norm of the coordinate projection of U to \mathbb{R}^n for $n < \nu$. Then rU is uniform on the sphere of radius r , and $r\rho_n$ is the norm of the projection of rU in n dimensions. Note that the random variable $r\rho_n$ increases with r . For the symmetric group, an analogous property in the form of *stochastic monotonicity* of $|f_{\nu \downarrow n}(\mathfrak{U}_{\nu, r})|$ in r holds for all three metrics we consider here (cf. Lemmas 3.6, 4.2, 5.2). We use the property as a key tool to identify the extreme ISS virtual permutations.

3 Hamming spherical symmetry

3.1 Classification theorem

Hamming distance between two permutations π and σ in S_n is defined as the number of positions $j \in [n]$ where $\pi(j) \neq \sigma(j)$. The distance to the identity permutation is $|\pi| = n - F(\pi)$, where $F(\pi) := \#\{j \in [n] : \pi(j) = j\}$ is the number of fixed points in π . Therefore, a sphere in the Hamming distance can have radius $0, 2, 3, \dots, n$. The sphere of radius 0 has a single element, which is the identity permutation. The elements of the sphere of radius n have no fixed points, they are called *derangements*. Thus a random permutation Π is Hamming-spherically symmetric if its distribution is conditionally uniform given the number of fixed points.

Let $f_n : S_n \rightarrow S_{n-1}$ be the operation of deleting element n from permutation written in the cycle notation. Then permutation $\sigma \in S_{n-1}$ has n extensions obtained by either inserting element n in a cycle next to the right of one of the elements in σ , or appending a singleton cycle (n) . For instance, five extensions of $(13)(24) \in S_4$ are

$$(153)(24), (135)(24), (13)(254), (13)(245), (13)(24)(5).$$

If a permutation $\sigma \in S_{n-1}$ has k fixed points, then it has 1 extension with $k + 1$ fixed points, k extensions with $k - 1$ fixed points and $n - k - 1$ extensions with k fixed points. Hence, the Hamming spherical symmetry is consistent under this system of projections.

The virtual permutations defined via the f_n 's were introduced in [19]. Writing permutation in the cycle notation yields a composite combinatorial structure, comprised of a partition of the set $[n]$ into disjoint nonempty blocks and a linear order on each block of the partition, with the property that in each block the smallest integer is also the minimal element of the order. Similarly, a virtual permutation corresponds to a partition of \mathbb{N} into some collection of disjoint nonempty blocks, taken together with a linear order on each block, such that within the block the smallest integer is also the minimal element in the linear order.

The sequential construction of uniform virtual permutation Π^* specializes in the cycle notation as the Dubins-Pitman *Chinese restaurant process* [16] with the following dynamics: given the permutation at step $n - 1$ is $\Pi_{n-1} = \sigma$, element n is inserted with probability $1/n$ in a cycle next to the right of any given element of σ , and appended to σ as singleton cycle (n) with probability $1/n$. This process has been intensely studied, in particular, it is well known that the random series comprised of the asymptotic frequencies of blocks follows the Poisson-Dirichlet/GEM distribution with parameter 1 [2, 16]. To this we only add here that Π^* can be seen as a partition of \mathbb{N} in infinitely many blocks, with the set of elements within each block ordered like the set of nonnegative rational numbers.

We introduce next a family of ISS virtual permutations $\{\Pi^\alpha, \alpha \in [0, 1]\}$, which includes the uniform virtual permutation as the $\alpha = 0$ case. Another edge case, $\alpha = 1$, corresponds to the trivial virtual permutation, which restricts to every $[n]$ as the identity.

For $\alpha \in (0, 1)$, to construct the virtual permutation Π^α explicitly, it will be convenient to introduce enriched permutations of $[n]$, which have an additional feature that genuine singleton cycles are distinguished from the cycles which will be bigger within a larger context $[\nu] \supset [n]$ but have a sole representative in $[n]$. For π a virtual permutation,

call element n *singular* if (n) is a singleton cycle in every π_ν , $\nu \geq n$, and call n *regular* otherwise. Define enriched virtual permutation $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2, \dots)$ to be a virtual permutation $\pi = (\pi_1, \pi_2, \dots)$ with additional classification of the elements in each π_n in singular and regular, and such that $\tilde{\pi}_n$ is consistent under the deletion of elements from cycles. For instance $\tilde{\pi}_6 = (13)(2)(46)(5)$ with singular element 2 and regular other five elements encodes that some elements $\nu > 6$ will be inserted in the cycle of every element except 2. The correspondence between π and $\tilde{\pi}$ is a canonical bijection, but for any fixed n , $\tilde{\pi}_n$ as compared to π_n contains more information about π .

For $\alpha \in [0, 1)$ define a random enriched virtual permutation by the following two rules.

- (i) Each n independently of other elements is singular with probability α .
- (ii) The virtual permutation restricted to the set of regular elements is distributed like the uniform Π^* , provided the regular elements are enumerated in increasing order by \mathbb{N} .

Here is a sequential construction of $\tilde{\Pi}^\alpha$, modifying the Chinese restaurant process. Element 1 is singular with probability α . Inductively, for $n > 1$, as an enriched permutation of $[n - 1]$ with some s singular fixed points (hence $n - s - 1$ regular elements) has been constructed, element n becomes singular with probability α , is inserted in existing cycle next to the right of any given regular element with probability $(1 - \alpha)/(n - s)$, and is appended as a regular singleton cycle with probability $(1 - \alpha)/(n - s)$.

Discarding the division into regular and singular elements yields Π^α . Explicitly, for the probability $p_{n,k} := \mathbb{P}(\Pi_n^\alpha = \sigma)$ of a permutation $\sigma \in S_n$ with k fixed points we have

$$p_{n,k}(\alpha) = \sum_{j=0}^k \binom{k}{j} \alpha^j (1 - \alpha)^{n-j} \frac{1}{(n-j)!}, \quad k \in \{0, 1, \dots, n-2, n\}. \quad (3.1)$$

The formula follows by noting that the probability of any given enriched permutation with j singular elements is $\alpha^j (1 - \alpha)^{n-j} / (n-j)!$, and that any j out of k fixed points can be singular. It is obvious from (3.1) that Π^α is ISS.

Lemma 3.1. *The virtual permutation Π^α satisfies*

$$\lim_{n \rightarrow \infty} \frac{F(\Pi_n^\alpha)}{n} \rightarrow \alpha$$

almost surely.

Proof. For any virtual permutation

$$\mathbb{P}(F(\pi_{n-1}) = k - 1 | F(\Pi_n) = k) = \frac{k}{n}, \quad \mathbb{P}(F(\pi_{n-1}) \geq k | F(\Pi_n) = k) = \frac{n - k}{n},$$

which readily implies that the sequence $F(\Pi_n)/n$ is a reverse submartingale, hence converges almost surely. For Π^* we have $\mathbb{E}[F(\Pi_n^*)] = 1$ hence the limit fraction is 0. The limit for $\alpha \neq 0$ follows from this and the construction of Π^α . \square

Theorem 3.2. *The extreme Hamming-ISS virtual permutations are $\{\Pi^\alpha, \alpha \in [0, 1]\}$.*

In the sequel we give two proofs of Theorem 3.2 using different techniques. The first proof in Section 3.2 is based on the explicit enumeration of spheres. The second proof in Section 3.3 uses stochastic monotonicity.

3.2 Proof through exact enumeration

A starting point for a straight approach to extreme virtual permutations is the enumeration of Hamming spheres. The sphere of radius n is the set of *derangements*, which are permutations with no fixed points. The number of derangements d_n is given by

$$d_n = n! \sum_{j=0}^n \frac{(-1)^j}{j!} = \left\lfloor \frac{n!}{e} + \frac{1}{n} \right\rfloor. \quad (3.2)$$

The first part of this formula is the classics due to de Montmort (1713), and the second is found in [17]. Denoting $D_{n,k}$ the number of permutations of $[n]$ with k fixed points, we have

$$D_{n,k} = \binom{n}{k} d_{n-k}. \quad (3.3)$$

Note that the elements comprising singleton cycles are precisely the fixed points of permutation. If $F(\sigma) = k$ for $\sigma \in S_{n-1}$ then σ has $n - k - 1$ extensions $\pi \in S_n$ with $F(\pi) = k$, one extension with $F(\pi) = k + 1$ and k extensions with $F(\pi) = k - 1$. Reciprocally, enumerating projections for given $\pi \in S_n$ with $F(\pi) = k$ yields the recursion

$$D_{n,k} = (n - k - 1)D_{n-1,k} + D_{n-1,k-1} + (k + 1)D_{n-1,k+1}, \quad 0 \leq k \leq n, \quad (3.4)$$

where $D_{1,0} = 0, D_{1,1} = 1$ and we adopt the convention $D_{n,j} = 0$ for $j \in \{-1, n + 1\}$. The recursion implies $D_{n,n-1} = 0$, in accord with the fact that permutation of $[n]$ cannot have $n - 1$ fixed points. There is some similarity between (3.4) and a two-term recursion for the Eulerian numbers counting descents [11], but these have very different properties.

For $\mathbf{\Pi} = (\Pi_1, \Pi_2, \dots)$ a ISS virtual permutation, let $p_{n,k} = \mathbb{P}(\Pi_n = \pi)$ be the probability of any given permutation $\pi \in S_n$ with $F(\pi) = n - |\pi| = k$ fixed points. We call the bivariate array $\mathbf{p} = (p_{n,k})$ the *probability function*. By the rule of addition of probabilities

$$p_{n,k} = (n - k)p_{n+1,k} + p_{n+1,k+1} + kp_{n+1,k-1}, \quad k \in \{0, 1, \dots, n - 2, n\}, \quad (3.5)$$

which is a backward recursion, dual to (3.4). Every nonnegative solution to (3.5) with $p_{1,1} = 1$ is a probability function for some unique random virtual permutation. Observe that for n fixed, $D_{n,k}p_{n,k}, 0 \leq k \leq n$, is the distribution of the number of fixed points $F(\Pi_n) = n - |\Pi_n|$, in particular

$$\sum_{k=0}^n D_{n,k}p_{n,k} = 1,$$

and $D_{n,0}p_{n,0} = d_n p_{n,0}$ is the probability that Π_n is a derangement.

Now we wish to explore the limit distributions which can arise in Theorem 2.1. To that end, for integer ν and $0 \leq \varkappa < \nu - 1$ let

$$p_{n,k}^{\nu,\varkappa} = \mathbb{P}(\Pi_n^* = \pi \mid F(\Pi_\nu^*) = \varkappa), \quad \text{where } \pi \in S_n, F(\pi) = k. \quad (3.6)$$

In terms of the Markov chain $(F(\Pi_n^*), n = 1, 2, \dots)$

$$D_{n,k}p_{n,k}^{\nu,\varkappa} = \mathbb{P}(F(\Pi_n^*) = k \mid F(\Pi_\nu^*) = \varkappa)$$

is the backward transition probability (also for any other ISS virtual permutation in place of Π^*). Viewed as a function of n and k , $p_{n,k}^{\nu,\varkappa}$ is an incomplete probability function which satisfies (3.5) for $n < \nu$ together with the boundary conditions $p_{\nu,\varkappa}^{\nu,\varkappa} = 1/D_{\nu,\varkappa}$ and $p_{\nu,k}^{\nu,\varkappa} = 0$ for $k \neq \varkappa$.

Let $D_{n,k}^{\nu,\varkappa}$ be the number of extensions of a given $\sigma \in S_n$ with $F(\sigma) = k$ to any $\pi \in S_\nu$ with $F(\pi) = \varkappa$. Then

$$p_{n,k}^{\nu,\varkappa} = \frac{D_{n,k}^{\nu,\varkappa}}{D_{\nu,\varkappa}}, \quad 1 \leq n \leq \nu, \quad (3.7)$$

where the ratio is called the *Martin kernel*. To find probability functions appearing as limits of the Martin kernel one needs to identify the regimes for $\varkappa = \varkappa(\nu)$ which ensure convergence of (3.7) as $\nu \rightarrow \infty$, for all fixed n and k . In the case of convergence, we will say that the limit probability function is *induced* by $\varkappa(\nu)$ as $\nu \rightarrow \infty$. The following simple observation allows us to only focus on the probabilities of derangements.

Lemma 3.3. *Every solution to (3.5) is uniquely determined by $(p_{n,0}, n = 1, 2, \dots)$.*

Proof. Re-write the recursion as $p_{n+1,k+1} = p_{n,k} - (n-k)p_{n+1,k} - kp_{n+1,k-1}$, and the conclusion is obvious by induction in k . \square

Now, $D_{n,0}^{\nu,\varkappa}$ counts extensions of a given derangement $\sigma \in S_n$ to a permutation $\pi \in S_\nu$ with \varkappa fixed points. Clearly, $D_{n,0}^{\nu,\varkappa} = 0$ if $\varkappa > \nu - n$. Otherwise, out of $\nu - n$ elements added to σ some $m \leq \nu - n - \varkappa$ elements, say $a_1 < \dots < a_m$ are allocated within the cycles present in σ , and there are $n(n+1) \dots (n+m-1)$ such allocations. The other $\nu - n - m$ elements $\{n+1, \dots, \nu\} \setminus \{a_1, \dots, a_m\}$ form new cycles of which \varkappa are singletons. Thus using (3.3) we obtain

$$\begin{aligned} D_{n,0}^{\nu,\varkappa} &= \sum_{m=0}^{\nu-n-\varkappa} \binom{\nu-n}{m} \frac{(n+m-1)!}{(n-1)!} D_{\nu-n-m,\varkappa} = \\ &= \sum_{m=0}^{\nu-n-\varkappa} \binom{\nu-n}{m} \frac{(n+m-1)!}{(n-1)!} \binom{\nu-n-m}{\varkappa} d_{\nu-n-m-\varkappa} = \\ &= \frac{(\nu-n)!}{\varkappa!} \sum_{m=0}^{\nu-n-\varkappa} \binom{n+m-1}{m} \frac{d_{\nu-n-m-\varkappa}}{(\nu-n-m-\varkappa)!}. \end{aligned}$$

Lemma 3.4. *As $\nu \rightarrow \infty$, the ratios (3.7) converge for $k = 0$ (hence also for $k \geq 0$) if and only if $\varkappa/\nu \rightarrow \alpha$ for some $\alpha \in [0, 1]$. In this case*

$$p_{n,0}^{\nu,\varkappa} \rightarrow \frac{(1-\alpha)^n}{n!}. \quad (3.8)$$

Proof. Applying (3.2) and (3.3) and using the above calculation for $D_{n,0}^{\nu,\varkappa}$, after cancellations we get as $\nu \rightarrow \infty$

$$p_{n,0}^{\nu,\varkappa} = \frac{D_{n,0}^{\nu,\varkappa}}{D_{\nu,\varkappa}} \sim \frac{1}{\nu^n} \sum_{m=0}^{\nu-n-\varkappa} \binom{n+m-1}{m} \sim \int_0^{1-\varkappa/\nu} \frac{x^{n-1}}{(n-1)!} dx = \frac{(1-\varkappa/\nu)^n}{n!},$$

thus the convergence holds if and only if $\varkappa/\nu \rightarrow \alpha$ for some $\alpha \in [0, 1]$. \square

We see that there exists a unique probability function $\mathbf{p}(\alpha) = (p_{n,k}(\alpha))$, with $p_{n,0}(\alpha) = (1 - \alpha)^n/n!$. Comparing with (3.1) we conclude that $\mathbf{p}(\alpha)$ is the probability function of the virtual permutation Π^α .

Proof of Theorem 3.2. Since $\mathbf{p}(\alpha)$'s (corresponding to Π^α) are the only possible limits in the context of Theorem 2.1, the family contains all extremes. On the other hand, by Lemma 3.1, $F(\Pi_n^\alpha)/n \rightarrow \alpha$ a.s. Thus the supports of distributions corresponding to different α are disjoint, and each $\mathbf{p}(\alpha)$ is extreme. \square

3.3 Proof through monotonicity

For the second proof we recall the notion of stochastic order.

Definition 3.5. For real random variables ξ and η we say that ξ is stochastically larger than η , denoted $\xi \geq_{\text{st}} \eta$ if either of the two equivalent properties hold:

- (a) For each $x \in \mathbb{R}$, we have $\mathbb{P}(\xi \geq x) \geq \mathbb{P}(\eta \geq x)$,
- (b) $\mathbb{E}[u(\xi)] \geq \mathbb{E}[u(\eta)]$ for every nondecreasing function u .

If $\xi \geq_{\text{st}} \eta$, then it is possible to define distributional copies of these variables, say ξ' and η' , on the same probability space in such a way that $\xi' \geq \eta'$ almost surely.

Lemma 3.6. Let Π and Π' be two random Hamming-spherically symmetric permutations in S_ν , such that $F(\Pi) \geq_{\text{st}} F(\Pi')$. Then also $F(f_{\nu \downarrow n}(\Pi)) \geq_{\text{st}} F(f_{\nu \downarrow n}(\Pi'))$ for $n < \nu$.

Proof. It is sufficient to prove the relation for $n = \nu - 1$, with $F(\Pi)$ and $F(\Pi')$ some given nonrandom values. Excluding the trivial case $F(\Pi) = \nu$ of the identity permutation, we further reduce to the case $F(\Pi') = k, F(\Pi) = k + 1$ with $0 \leq k < \nu - 2$. The general case will follow by induction and using the fact that the stochastic order is preserved by convex mixtures.

We have $F(f_\nu(\Pi)) \in \{k, k + 1, k + 2\}$ and $F(f_\nu(\Pi')) \in \{k - 1, k, k + 1\}$, thus to show that $F(f_\nu(\Pi))$ is stochastically larger than $F(f_\nu(\Pi'))$ we only need to check that

$$\mathbb{P}(F(f_\nu(\Pi')) = k + 1) \leq \mathbb{P}(F(f_\nu(\Pi)) \in \{k + 1, k + 2\}). \quad (3.9)$$

Since Π is uniformly distributed over permutations with $k + 1$ fixed points, and since the event in the right-hand side of (3.9) occurs precisely when ν is not a fixed point, the probability of this event is $(\nu - k - 1)/\nu$. Likewise, the event on the left-hand side implies that ν is neither a fixed point nor belongs to a $(\nu - k)$ -cycle of Π' . The probability that ν is a fixed point of Π' is k/ν . The probability that Π' has a $(\nu - k)$ -cycle containing ν is

$$\frac{(\nu - k)}{\nu} \frac{(\nu - k - 1)!}{d_{\nu - k}} < \frac{(\nu - k)}{\nu} \frac{(\nu - k - 1)!}{(\nu - k)!} = \frac{1}{\nu},$$

where we used the obvious bound $d_{\nu - k} < (\nu - k)!$ along with the fact that given the cycle structure ν is equally likely to occupy any position within the cycles. Now (3.9) follows:

$$\mathbb{P}(F(f_\nu(\Pi')) = k + 1) \leq 1 - \frac{k}{\nu} - \frac{1}{\nu} = \frac{\nu - k - 1}{\nu} = \mathbb{P}(F(f_\nu(\Pi)) \in \{k + 1, k + 2\}).$$

\square

Proof of Theorem 3.2. Let $\mathbf{\Pi} = (\Pi_n, n = 1, 2, \dots)$ be extreme Hamming-ISS virtual permutation. There exists a sequence $\varkappa(\nu)$ such that $\mathbf{\Pi}$ is representable as a limit of $\mathfrak{U}_{\nu, \nu - \varkappa(\nu)}$ as in Theorem 2.1 (where $r(\nu) = \nu - \varkappa(\nu)$). Passing if necessary to a subsequence we may assume that $\varkappa(\nu)/\nu \rightarrow \alpha$ as $\nu \rightarrow \infty$ for some $\alpha \in [0, 1]$.

Suppose first $\alpha \in (0, 1)$ and choose $0 < \epsilon < \min(\alpha, 1 - \alpha)$. As $\nu \rightarrow \infty$, $F(f_{\nu \downarrow n} \mathfrak{U}_{\nu, \nu - k(\nu)})$ converges in distribution to $F(\Pi_n)$, and by Lemma 3.1 the probability of relation $F(\Pi_n^{\alpha+\epsilon}) > \varkappa(\nu)$ approaches 1. Likewise, the probability of relation $F(\Pi_n^{\alpha-\epsilon}) < \varkappa(\nu)$ approaches 1. Invoking Lemma 3.6 we obtain for projections

$$F(\Pi_n^{\alpha-\epsilon}) \leq_{\text{st}} F(\Pi_n) \leq_{\text{st}} F(\Pi_n^{\alpha+\epsilon}).$$

By continuity in the parameter both bounds converge in distribution to $F(\Pi_n^\alpha)$ as $\epsilon \rightarrow 0$. Thus $F(\Pi_n)$ has the same distribution as $F(\Pi_n^\alpha)$, implying that Π_n and Π_n^α have the same distribution for every n . It follows that $\mathbf{\Pi}$ has the same distribution as $\mathbf{\Pi}^\alpha$.

The edge cases $\alpha \in \{0, 1\}$ are treated similarly, with one-sided bounds derived from Π_n^ϵ and $\Pi_n^{1-\epsilon}$, respectively.

It follows that virtual permutations $\{\mathbf{\Pi}^\alpha, \alpha \in [0, 1]\}$ are the only possible limits in Theorem 2.1. Since their supports are disjoint by Lemma 3.1, this is the set of extremes. \square

3.4 Complements

1. For $\tilde{\mathbf{\Pi}}^\alpha$ the bivariate process counting singular and regular fixed points is a Markov chain with transition probabilities at step n being

$$(s, r) \rightarrow \begin{cases} (s+1, r) & \text{w.p. } \alpha \\ (s, r+1) & \text{w.p. } \frac{1-\alpha}{n-s} \\ (s, r-1) & \text{w.p. } \frac{(1-\alpha)r}{n-s} \\ (s, r) & \text{w.p. } \frac{(1-\alpha)(n-1-r-s)}{n-s}. \end{cases}$$

For the count of fixed points $F(\Pi_n^\alpha)$ the transition probabilities are more involved.

2. With ISS virtual permutation $\mathbf{\Pi}$ one can uniquely associate a partition of the infinite set \mathbb{N} , by assigning integers i and j to the same block if they belong to the same cycle of Π_n for $n \geq \max(i, j)$. This partition is exchangeable, that is has distribution invariant under bijections $\mathbb{N} \rightarrow \mathbb{N}$ moving finitely many elements. Partitions with nonzero frequency of singletons (dust component) appear as intermediate states in exchangeable coalescence processes (e.g. [14]).

The distribution of exchangeable partition is determined by the exchangeable partition probability function (EPPF), see [16]. Our classification of ISS virtual permutations can be recast as the characterisation of partitions of \mathbb{N} with EPPF of the form

$$p(n_1, \dots, n_\ell) = p_{n,k} \prod_{j=1}^{\ell} (n_j - 1)!,$$

where n_1, \dots, n_ℓ is a partition of integer n and k is the multiplicity of part 1.

3. It is well known that for the uniform Π_n^* , the sequence of cycle sizes arranged in non-increasing order and normalised by n converges weakly to the Poisson-Dirichlet distribution with parameter 1 [2, 16]. The same limit holds for Π_n^α , provided the cycle sizes are normalised by $(1 - \alpha)n$.

4 Kendall-tau spherical symmetry

4.1 Inversions in permutation

The Kendall-tau distance between π and σ in S_n is the number of discordant pairs, i.e. positions $i < j$ with $\text{sgn}(\pi(i) - \pi(j)) = -\text{sgn}(\sigma(i) - \sigma(j))$. When σ is the identity permutation, discordant pair $i < j$ is inversion in π , thus $|\pi|$ coincides with the number of inversions

$$I(\pi) := \#\{(i, j) : 1 \leq i < j \leq n : \pi(i) > \pi(j)\}.$$

In this setting spherical symmetry means that permutations of $[n]$ with the same number of inversions have equal probability.

Viewing permutation $\pi \in S_n$ as a linear order on the set of *positions* and using the one-line notation $(\pi(1), \dots, \pi(n))$, we understand $\pi(j)$ as the rank of element j among $[n]$ (so $\pi(j) = 1$ if j is the minimal element in the order). It is useful to observe that for the inverse permutation $I(\pi^{-1}) = I(\pi)$, which suggests two systems of projections, each consistent with the spherical symmetry:

- (i) $f'_n(\pi)$ deletes the last entry $\pi(n)$, and re-labels $\pi(1), \dots, \pi(n-1)$ by an increasing bijection with $[n-1]$.
- (ii) $f''_n(\pi)$ deletes letter n from the one-line notation.

For instance,

$$f'_5 : (2, 5, 1, 4, 3) \mapsto (2, 4, 1, 3), \quad f''_5 : (2, 5, 1, 4, 3) \mapsto (2, 1, 4, 3).$$

The projections are mapped into one another by the group inversion.

For convenience we will work with projection f'_n , which may be also seen as the restriction of order from $[n]$ to $[n-1]$. The advantage of this choice of projection is that the set of inversions within $[n]$ remains unaltered as the permutation gets extended. Furthermore, the mapping $(\pi(1), \dots, \pi(n)) \mapsto (n - \pi(1), \dots, n - \pi(n))$ yields the inverse order relation, hence also consistent with the projections.

A virtual permutation $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots)$ in this setting is a system of consistent orders on $[n]$ for $n = 1, 2, \dots$, hence defines a linear order on \mathbb{N} . Note that $\pi_{n+1}(j) - \pi_n(j)$ is 1 or 0 depending on whether $\pi_{n+1}(n+1) \leq j$ or not. The order defined by $\boldsymbol{\pi}$ is a well-order (i.e. isomorphic to (\mathbb{N}, \leq)) if and only if $\pi_n(j)$ has a terminal value as n increases, for every j .

Observe that $I(\pi) = \sum_{j=1}^n \eta_j$, where $\eta_j := \#\{i : 1 \leq i < j, \pi(i) > \pi(j)\}$. The mapping $\pi \mapsto \eta_1, \dots, \eta_n$ is a bijection called the *Lehmer code*. For instance, $(2, 5, 1, 4, 3)$ is encoded into $0, 0, 2, 1, 2$. In terms of the Lehmer code, f'_n acts as the coordinate projection which

sends $\eta_1, \dots, \eta_{n-1}, \eta_n$ to $\eta_1, \dots, \eta_{n-1}$. The consistency with the Kendall-tau distance is now easily seen. Indeed, if $\pi \in S_n$ is an extension of $\sigma \in S_{n-1}$, the counts of inversions are related as $I(\pi) = I(\sigma) + n - \pi(n) = I(\sigma) + \eta_n$. The idea of the Lehmer code generalises to the infinite setting, allowing us to encode the order on \mathbb{N} in a single string η_1, η_2, \dots .

4.2 Mallows distributions

For the virtual permutation Π^* the terms of the Lehmer code η_1, η_2, \dots are independent, with η_j being uniformly distributed on $\{0, 1, \dots, j-1\}$. This connects the problem of classification of the ISS permutations to the study of conditional laws and tail algebras for sums of independent random variables, see [1, 20] and especially [3].

Exponential tilting of the uniform distribution yields a truncated geometric distribution with masses $q^i/[j]_q$ ($0 \leq i \leq j-1$), where $[j]_q := (1 - q^j)/(1 - q)$. The tilted joint distribution of η_1, \dots, η_n conditional on $\eta_1 + \dots + \eta_n$ does not depend on q , hence the corresponding permutation is Kendall-tau spherically symmetric, with distribution

$$\mathbb{P}(\Pi_n = \pi) = \frac{q^{I(\pi)}}{[n]_q!}, \quad \pi \in S_n, \quad (4.1)$$

where $[n]_q! := \prod_{j=1}^n [j]_q$. This is the *Mallows distribution* on permutations. The parameter range is $q \in [0, \infty]$, where the edge cases 0 and ∞ correspond to the deterministic identity and the decreasing permutation $(n, n-1, \dots, 1)$, respectively.

Under the Mallows distribution with $q < 1$ the virtual permutation determines a well-order on \mathbb{N} . Passing to the inverse order relation yields a Mallows distribution with parameter $1/q$. See [5, 13] for further properties of the Mallows permutations.

Theorem 4.1. *Mallows distributions (4.1) with $q \in [0, \infty]$ and only they are the extreme ISS virtual permutations with respect to the Kendall-tau distance.*

The proof hinges on the following analogue of Lemma 3.6.

Lemma 4.2. *Let Π and Π' be two random Kendall-tau-spherically symmetric permutations in S_ν , such that $I(\Pi) \geq_{\text{st}} I(\Pi')$. Then also $I(f'_{\nu \downarrow n}(\Pi)) \geq_{\text{st}} I(f'_{\nu \downarrow n}(\Pi'))$ for $n < \nu$.*

To show this we will need the following property of the uniform distribution.

Lemma 4.3. *For $i = 1, 2, 3$ let (U_i, V_i) be pairs of integer random variables such that*

- (i) V_1 and U_1 are independent, with U_1 uniformly distributed on some integer interval,
- (ii) the conditional distribution of (V_i, U_i) given $V_i + U_i$ is the same for $i = 1, 2, 3$,
- (iii) $V_2 + U_2 \geq_{\text{st}} V_3 + U_3$.

Then $V_2 \geq_{\text{st}} V_3$.

Proof. It is sufficient to consider the case with U_1 uniformly distributed on $\{1, 2, \dots, k\}$, $V_2 + U_2 = k + 1$ and $V_3 + U_3 = k$, where k is some constant. The general case will follow by shifting the range of the variables, induction and taking mixtures.

Then for $1 \leq m \leq k$ we have

$$\begin{aligned}
\mathbb{P}(V_2 \geq m) &= \mathbb{P}(V_1 \geq m | V_1 + U_1 = k + 1) = \\
&= \frac{\sum_{j=m}^k \mathbb{P}(V_1 = j, U_1 = k + 1 - j)}{\sum_{j=1}^k \mathbb{P}(V_1 = j, U_1 = k + 1 - j)} = \frac{k^{-1} \mathbb{P}(m \leq V \leq k)}{k^{-1} \mathbb{P}(1 \leq V \leq k)} = \\
&= \frac{\mathbb{P}(m \leq V_1 \leq k - 1) + \mathbb{P}(V_1 = k)}{\mathbb{P}(1 \leq V_1 \leq k - 1) + \mathbb{P}(V_1 = k)} \geq \frac{\mathbb{P}(m \leq V_1 \leq k - 1)}{\mathbb{P}(1 \leq V_1 \leq k - 1)} \geq \\
&= \frac{k^{-1} \mathbb{P}(m \leq V_1 \leq k - 1)}{k^{-1} \mathbb{P}(0 \leq V_1 \leq k - 1)} = \mathbb{P}(V_1 \geq m | V_1 + U_1 = k) = \mathbb{P}(V_3 \geq m),
\end{aligned}$$

where we used that $(a + x)/(b + x)$ increases in $x \geq 0$ for $b \geq a > 0$. The cases $m > k$ or $m < 1$ are trivial, and the relation follows. \square

Proof of Lemma 4.2. We apply Lemma 4.3 repeatedly to the Lehmer code of permutations Π_n^* , Π and Π' , respectively. Here, U_i is the last coordinate of the code and V_i is the sum of all other coordinates. \square

Proof of Theorem 4.1. The argument goes along the line of proof of Theorem 3.2 hence we omit some details. By the strong law of large numbers for sums, under the Mallows distribution (4.1) the number of inversions satisfies the almost sure asymptotics:

- (a) $I(\Pi_n) \sim \frac{q}{1-q} n$ for $0 \leq q < 1$,
- (b) $I(\Pi_n) \sim \frac{1}{4} n^2$ for $q = 1$ (the uniform case),
- (c) $\binom{n}{2} - I(\Pi_n) \sim \frac{q^{-1}}{1-q^{-1}} n$ for $1 < q \leq \infty$.

Let $\varkappa(\nu)$ be a sequence inducing an extreme ISS virtual permutation $\mathbf{\Pi} = (\Pi_n, n = 1, 2, \dots)$, as in Theorem 2.1 (so $r(\nu) = \varkappa(\nu)$). Passing to a subsequence of the values of ν we can achieve that either $\varkappa(\nu) \sim \frac{q}{1-q} \nu$ or $\binom{\nu}{2} - \varkappa(\nu) \sim \frac{q^{-1}}{1-q^{-1}} \nu$ for some q , or both $\varkappa(\nu)$ and $\binom{\nu}{2} - \varkappa(\nu)$ grow faster than linearly as $\nu \rightarrow \infty$.

Consider the case $\varkappa(\nu) \sim \frac{q}{1-q} \nu$ with $0 < q < 1$. Fix n and $0 < \varepsilon < \min(q, 1 - q)$. To construct stochastic upper and lower bounds for $I(\Pi_n)$ we use virtual permutations $\text{Mallows}(q \pm \varepsilon)$ and appeal to Lemma 4.2. Sending $\varepsilon \rightarrow 0$ we conclude that Π_n is $\text{Mallows}(q)$ for every n , hence $\mathbf{\Pi}$ is $\text{Mallows}(q)$.

Two other asymptotic regimes for $\varkappa(\nu)$ are treated similarly. It follows that the family of $\text{Mallows}(q)$ virtual permutations contains all extreme ISS virtual permutations. Since by (a), (b) and (c) they all have disjoint supports all these are extreme. \square

It is of interest to recast Theorem 4.1 in terms of ratios of combinatorial numbers. The number of permutations $\sigma \in S_n$ with k inversions is the *Mahonian number* $M_{n,k}$ counting solutions to the equation $\eta_1 + \dots + \eta_n = k$, where η_1, η_2, \dots are integer variables satisfying $0 \leq \eta_j < j$. The number of extensions of any such σ to a permutation $\pi \in S_\nu$ with $I(\pi) = \varkappa$ is a generalised Mahonian number $M_{n,k}^{\nu, \varkappa}$ counting solutions to $\eta_{n+1} + \dots + \eta_\nu = \varkappa - k$. Identifying

$$\frac{M_{n,k}^{\nu, \varkappa}}{M_{\nu, \varkappa}} \tag{4.2}$$

with the Martin kernel we obtain

Corollary 4.4. *If as $\nu \rightarrow \infty$ and $\varkappa = \varkappa(\nu)$ varies in some way the ratios (4.2) converge for all n and $0 \leq k \leq \binom{n}{2}$ then the limit is $q^k/[n]_q!$ for some $q \in [0, \infty]$. The convergence holds if and only if either $\varkappa(\nu) \sim \frac{q}{1-q}\nu$ for $0 \leq q < 1$, or $\binom{\nu}{2} - \varkappa(\nu) \sim \frac{q^{-1}}{1-q^{-1}}\nu$ for $1 < q \leq \infty$, or both $\varkappa(\nu)$ and $\binom{\nu}{2} - \varkappa(\nu)$ grow faster than linearly for $q = 1$.*

5 Cayley spherical symmetry

The Cayley distance between π and σ in S_n is defined as the minimal number of transpositions needed to transform one permutation in another. Right-multiplying $\pi \in S_n$ by the transposition (i, j) amounts to swapping letters i and j in the one-row notation of π . The multiplication increases the number of cycles by one if i and j belong to the same cycle of π , and decreases by one otherwise. Thus the distance to the identity is $|\pi| = n - C(\pi)$, where $C(\pi)$ denotes the number of cycles in π . We take the same projections f_n as in the setting with Hamming distance of Section 3. If $\sigma \in S_{n-1}$ is a permutation with k cycles, then it has $n - 1$ extensions with the same number of cycles and 1 extension with $k + 1$ cycles. Hence, the Cayley spherical symmetry is preserved under these projections.

For virtual permutation $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots)$, we have $C(\pi_n) = \sum_{j=1}^n \beta_j$ where $\beta_j = 1$ if j is a fixed-point of π_j and $\beta_j = 0$ otherwise. For the uniform $\boldsymbol{\Pi}^*$, the sequence of β_j 's is independent Bernoulli with $\mathbb{P}(\beta_j = 1) = 1/j$. Exponential tilting with parameter $\theta \in [0, \infty]$ yields a family of ISS virtual permutations with the *Ewens distribution*

$$\mathbb{P}(\Pi_n = \pi) = \frac{\theta^{C(\pi)}}{(\theta)_n}, \quad \pi \in S_n, \quad (5.1)$$

where $(\theta)_n = \prod_{i=0}^{n-1} (\theta + i)$. For $\theta = 0$ this is a uniform cyclic permutation, and for $\theta = \infty$ the unit mass at the identity. Under the Ewens distribution $\boldsymbol{\Pi}$ follows the dynamics of the Chinese restaurant process, where element n is a new cycle appended to Π_{n-1} with probability $\theta/(\theta + n - 1)$, and is inserted next to the right in the cycle of any $j \in [n - 1]$ with probability $1/(\theta + n - 1)$ [16].

Theorem 5.1. *The Ewens distributions (5.1) with $\theta \in [0, \infty]$ and only they are the extreme ISS virtual permutations with respect to the Cayley distance.*

A proof of Theorem 5.1 can be found in [12], where it appears in a minor disguise (see also [10, Theorem 4.1]). Following our unified approach, the key observation is the following lemma:

Lemma 5.2. *Let Π and Π' be two random Cayley-spherically symmetric permutations in S_ν , such that $C(\Pi) \geq_{\text{st}} C(\Pi')$. Then for every $n < \nu$, also $C(f_{\nu \downarrow n}(\Pi)) \geq_{\text{st}} C(f_{\nu \downarrow n}(\Pi'))$.*

Proof. This immediately follows from the fact that if $\pi \in S_n$ has k cycles, then $f_n(\pi)$ has either k or $k - 1$ cycles. \square

Using Lemma 5.2 and that under the Ewens distribution with $\theta \in (0, \infty)$ the number of cycles grows as $C(\Pi_n) \sim \theta \log n$, see e.g. [2], the proof of Theorem 5.1 can be produced along the same lines as in Theorems 3.2 and 4.1. A counterpart of Corollary 4.4 concerns limiting regimes for the ratios of generalised Stirling numbers of the first kind.

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